International Journal of Theoretical Physics, Vol. 32, No. 5, 1993

Generalization of Quantum Statistics in Statistical Mechanics

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Received June 20, 1992

We propose a generalization of quantum statistics in the framework of statistical mechanics. We derive a general formula which involves a wide class of equilibrium quantum statistical distributions, including the Bose and Fermi distributions. We suggest a way of evaluating the statistical distributions with the help of many-particle partition functions and apply it to studying some interesting distributions. A question on the statistical distribution for anyons is discussed, and the term following the Boltzmann one in the expansion of this distribution in powers of the Boltzmann factor, $\exp[\beta(\mu-\varepsilon_i)]$, is estimated. An ansatz is proposed for evaluating the statistical distribution for *quons* (particles whose creation and annihilation operators satisfy the *q-commutation relations*). We also treat non-equilibrium statistical mechanics, obtaining unified expressions for the entropy of a nonequilibrium quantum gas and for a collision integral which are valid for a wide class of statistics.

1. INTRODUCTION

In recent years interest has increased in studying quantum statistics different from Bose–Einstein and Fermi–Dirac statistics (Lee *et al.*, 1989; Fredenhagen *et al.*, 1989; Greenberg, 1990; Imbo *et al.*, 1990; Longo, 1990; Mohapatra, 1990; Balachandran, 1991; Chen and Ni, 1991; Aneziris *et al.*, 1991). There are two principal approaches to such statistics, which are realized in the framework of the first-quantized and second-quantized theories, respectively. We propose a third approach in the framework of statistical mechanics.

The first approach is based on studying the possible symmetry types of the state vectors of systems of identical particles with respect to permutations of their arguments. As is well known, Bose–Einstein and Fermi–Dirac statistics correspond to the totally symmetric and totally antisymmetric wave

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0020-7748/93/0500-0737$07.00/0

functions, that is, the symmetric and antisymmetric representations of the permutation group. Messiah and Greenberg (1964) have observed that the state vector of a system of N identical particles may have, in the general case, a symmetry corresponding to any irreducible representation of the permutation group S_N [this fact was pointed out by Dirac (1958)].

In the latter case, the symmetry of the N-particle state vector corresponding to some irreducible representation of S_N may lead to several different types of symmetry of the $(N-1)$-particle state vectors [due to the fact that the restriction to the subgroup S_{N-1} of any irreducible representation of S_N, other than symmetric or antisymmetric, contains several nonequivalent irreducible representations of S_{N-1}; see, e.g., Hamermesh (1962)]. In accordance with this, a general N-particle state vector may be a mixture of states with symmetries each determined by some irreducible representation of S_N (Hartle and Taylor, 1969; Stolt and Taylor, 1970) [see also related considerations in the framework of algebraic quantum field theory by Dopplicher *et al.* (1971, 1974). And what is more, the N-particle state vectors may have such a symmetry that these will not run over all the space spanned by the vectors realizing a given irreducible representation of S_N, but only over certain of its subspaces (Ohnuki and Kamefuchi, 1982). A complete classification of possible symmetry types of many-particle state vectors based on the permutation group treatment is an open problem.

The above consideration is valid only if particles move in three or more spatial dimensions. This fact has been established in terms of the quantization method, which takes into account the particle identity already in determinating the configuration space of a many-particle system. The configuration space for N identical particles Q_N, if these are spinless, is defined in this method as follows (Laidlaw and DeWitt, 1971):

$$Q_N=(M^N-D)/S_N \tag{1}$$

where M is the configuration space of one particle (for example, M is $\mathbb{R}^2$), D is the diagonal of the space M^N (the set of points of M^N for which the coordinates of at least two particles coincide). The quotient on S_N just reflects the fact that the configurations which are obtained from each other by means of some particle permutation correspond to the same physical state. The configuration space for N spinning identical particles is more complicated (Tscheuschner, 1989; Balachandran *et al.*, 1990).

For a two-dimensional single-particle configuration space, Q_N of (1) is multiply connected. The wave function determined on multiply connected configuration space is many-valued in the general case (Finkelstein and Rubinstein, 1968). A particle permutation corresponds to a closed path in Q_N. The set of closed nongomotopic (cannot be continuously deformed to each other) paths on a manifold composes the fundamental group π_1 of the manifold. Therefore, giving a certain change of the state vector for every particle permutation or, equivalently, for every closed path in Q_N means giving a certain representation of the group $\pi_1(Q_N)$. In accordance with this, the possible symmetry types of the many-particle state vectors determined on the configuration space Q_N are classified by irreducible representations of the fundamental group of this space $\pi_1(Q_N)$.

As noticed by Wu (1984*a*) and Ringwood and Woodward (1984), $\pi_1(Q_N)$ with Q_N from (1) is nothing but the *braid group* $B_N(M)$ of the manifold M. Thus, in two spatial dimensions, the possible state vector symmetry types of a system of spinless identical particles are classified by the braid group irreducible representations [see Imbo *et al.* (1990) for more rigorous formulations]. Recently an approach resulting in the classification of statistics based on the braid groups has also been formulated in the framework of algebraic quantum field theory (Fredenhagen *et al.*, 1989). On the fundamental group of the configuration space for the more involved case of spinning particles see Balachandran *et al.* (1991).

The first example of the braid group many-particle state vector symmetry was given by Leinaas and Myrheim (1977). They showed that the many-particle wave function determined on the configuration space (1) may have a symmetry such that it is multiplied by the phase factor $e^{i\theta}$, with θ in $0 \leq \theta < 2\pi$, under the permutation of any two coordinates. So, for two particles,

$$\Psi(2, 1) = e^{i\theta}\Psi(1, 2) \tag{2}$$

The symmetry (2) interpolates between those of the two-particle wave functions for the boson ($\theta = 0$) and fermion ($\theta = \pi$) systems.

The particles whose wave function has the symmetry (2), with $\theta \neq 0, \pi$, were called *anyons* and the corresponding statistics were called *θ-statistics* or *fractional* statistics. In recent years, θ-statistics has attracted great attention because of its use in two-dimensional physics: some quasiparticles appearing in theories of the fractional quantum Hall effect and high-temperature superconductivity are considered to obey this statistics (Halperin, 1984; Kalmeyer and Laughlin, 1987).

A complete classification of the braid group irreducible representations is lacking. Recently, as a result of significant progress on this issue, new types of the braid group irreducible representations have been found in mathematical studies (Jones, 1987; Wenzl, 1988; Birman and Wenzl, 1989) and in studies of exactly solvable models in statistical mechanics (Wadati *et al.*, 1989) which provide new examples of possible symmetry types of the identical-particle system state vectors (Lee *et al.*, 1989; Fredenhagen *et al.*, 1989; Longo, 1990).

Note that for the braid group statistics—like the statistics based on the permutation group—the symmetry of the N-particle state vector is not necessarily determined by one of the irreducible representations of the fundamental group of the configuration space of identical particles $\pi_1(Q_N)$ [$B_N(M)$ for spinless particles] but may be determined by several of them.

The second approach to generalization of statistics deals with the commutation relations for creation and annihilation operators more general than those for Bose and Fermi operators. This approach was initiated by Green (1953), who introduced the so-called *para-boson* and *para-fermion* commutation relations, which have been extensively studied [for review, see Ohnuki and Kamefuchi (1982)]. Okayama (1952) began constructing the commutation relations for the particles whose number in the same quantum state may take values from zero to d (we will call statistics which such particles obey *d-statistics*). Such relations have been derived by Kuryshkin (1988) for the case of systems with a single quantum state. More recently, in the context of studying possible small violations of the Pauli principle, the generalized commutation relations

$$a_i a_j^+ - q a_j^+ a_i = \delta_{ij} \tag{3}$$

with q in $-1 \leq q \leq 1$, have been suggested (Greenberg, 1990; Mohapatra, 1990) interpolating between the boson ($q=1$) and fermion ($q=-1$) ones. We will refer to (3) as the *q-commutation relations* and to the particles whose annihilation and creation operators satisfying (3) as *quons* [following Greenberg (1991)].

The present paper is devoted to an approach to the generalization of statistics which is carried out in the framework of statistical mechanics. The aim of this approach is to provide a framework for the discussion of statistics more general than Bose–Einstein and Fermi–Dirac ones. As for equilibrium statistical mechanics, one of the tasks of this approach is to introduce new statistical distributions among which one could, in particular, look for the statistical distributions corresponding to the statistics appearing in the two above approaches [note that a family of distributions other than the Bose and Fermi ones—corresponding to d-statistics—was derived by Gentile (1940)]. The following question arises here: what is the most general form for the quantum statistical distribution?

In this paper we present the first steps in the development of the statistical mechanical approach to the generalization of statistics and establish some correspondences with the two above approaches based on considerations of many-particle state vector symmetries and commutation relations. The rest of the paper is organized as follows.

In Section 2, supposing, as usual, that the quantum states are filled by particles independent of each other, but refusing any restrictions on allowed particle occupation numbers of a distinct state, we derive a general formula for the quantum statistical distribution

$$n_i = \frac{x_i \Xi_i'(x_i)}{\Xi_i(x_i)} \tag{4}$$

depending on an arbitrary [up to the condition of the existence of the Boltzmann limit for the distribution (4)] function $\Xi_i(x_i)$ with $x_i = \exp[\beta(\mu - \varepsilon_i)]$, $\beta = k_B T$. The formula (4) involves a wide class of statistical distributions, including the Bose and Fermi distributions.

Section 3 deals with statistical distributions which allow an expansion in a power series in x_i:

$$n_i = x_i + a_2 x_i^2 + a_3 x_i^3 + \cdots \tag{5}$$

which is closely related to the virial expansion for a quantum gas of free particles. We express the coefficients $a_2, a_3, \ldots,$ in terms of the many-particle partition functions of noninteracting particles, which provides a way of finding the statistical distribution when the latter are known.

In Sections 4 and 5 the general considerations of the preceding sections are applied to concrete statistical distributions. Section 4 discusses the question of the statistical distribution for anyons. We use the results of Arovas *et al.* (1985) to evaluate the second term in the expansion (5) for anyons. In Section 5 we propose ansatz which enables one to determine the statistical distribution for quons. To conclude that section, we introduce some generalizations of the q-commutation relations for single-energy-level systems leading to new statistical distributions.

Section 6 is devoted to a unified description of nonequilibrium gases consisting of particles obeying any statistics belonging to the class (4). A general expression for a nonequilibrium quantum gas entropy is derived which reduces to those for bosons and fermions in special cases. In addition, with the aid of a generalization of the Einstein relations for the absorption and emission probabilities, a general form for a collision integral for such particles is obtained.

2. A CLASS OF QUANTUM STATISTICAL DISTRIBUTIONS

We begin with a short review of a common way of deriving quantum statistical distributions (see, e.g., Landau and Lifshitz, 1980). This starts from the expression for the grand partition function of a quasiclosed subsystem resulting from the Gibbs distribution,

$$\Xi = \sum_{\mathcal{N}} \sum_{k} \exp[\beta(\mu \mathcal{N} - E_{\mathcal{N},k})] \tag{6}$$

where the index k labels the states of the subsystem for a given particle number $\mathcal{N}$ in it, and $E_{\mathcal{N},k}$ are the energies of these states. Next, a single-particle quantum state i of energy ε_i is considered, and the particles in this state are chosen as a quasiclosed subsystem. The total particle energy in such a subsystem is

$$E_{N_i} = N_i \varepsilon_i \tag{7}$$

where N_i is the particle number in the ith state. With the use of (7), (6) leads to the following expression for the grand partition function for the particles in the ith state:

$$\Xi_i = \sum_{N_i} \{e^{\beta(\mu - \varepsilon_i)}\}^{N_i} \tag{8}$$

The expressions for the partition functions for fermions or bosons are obtained from (8) if one considers that N_i may take values 0, 1 or 0, 1, 2, . . . , respectively. If one allows the occupation numbers N_i to take values 0, 1, . . . , d, where d is a positive integer (in other words, one considers that not more than d particles may be in the same quantum state), then (8) results in the partition function for particles obeying d-statistics (Okayama, 1952):

$$(\Xi_i)_d = \sum_{l=0}^{d} (x_i)^l \tag{9}$$

with

$$x_i = e^{\beta(\mu - \varepsilon_i)} \tag{10}$$

The statistical distribution, being the average particle number in state i, is found with the aid of the thermodynamic identities

$$n_i = -(\partial \Omega_i / \partial \mu)_T, \qquad \Omega_i = -\beta^{-1} \ln \Xi_i \tag{11}$$

For instance, Ξ_i of (9) results in the statistical distribution for particles obeying d-statistics (Gentile, 1940):

$$(n_i)_d = \left\{ \sum_{l=0}^{d} (x_i)^l \right\}^{-1} \sum_{l=0}^{d} l (x_i)^l \tag{12}$$

interpolating between the Fermi ($d = 1$) and Bose ($d \to \infty$) distributions.

So we see that the particular assumptions concerning the allowed values of the occupation numbers N_i lead to concrete statistical distributions. To obtain a general formula for the statistical distribution, we make no assumption concerning the possible occupation numbers. At the same time, we will assume as above that the particles in the same quantum state can be regarded as a quasiclosed subsystem, which means that the quantum states are filled by particles independent of each other (note that the more general situation is logically possible; see Section 7).

We observe that, according to (8), Ξ_i depends on the quantities ε_i, μ, β only through the combination x_i of (10), that is, Ξ_i is a certain function of x_i:

$$\Xi_i = \Xi_i(x_i) \tag{13}$$

Then, using (11), we obtain the statistical distribution

$$n_i = \frac{x_i \Xi_i'(x_i)}{\Xi_i(x_i)} \tag{14}$$

where the prime indicates differentiation with respect to x_i.

There exists a restriction to the function $\Xi_i(x_i)$ which appears as the condition of the existence of the Boltzmann limit for the statistical distribution. Namely, in the limit $\beta\mu \to -\infty$, that is, according to (10), $x_i \to 0$, the distribution (14) should tend to the Boltzmann distribution x_i. This gives the constraints

$$\Xi_i'(x_i)/\Xi_i(x_i) \to 1 \qquad \text{as} \quad x_i \to 0 \tag{15}$$

Supposing the condition (15) to be fulfilled, we regard the function $\Xi_i(x_i)$ as arbitrary in other respects.

The expression (14) is a general form for the statistical distribution for the case when each quantum state is filled by particles independent of filling the other states.

The formula (14) involves a wide class of statistical distributions parametrized by the function $\Xi_i(x_i)$. In special cases, (14) reduces to known statistical distributions. So, for $\Xi_i(x_i)$ equal to $(1+x_i)$ and $(1-x_i)^{-1}$, (14) coincides with the Fermi and Bose distributions, respectively. For the partition function (9), equation (14) results in the statistical distribution for particles obeying d-statistics (12). The class (14) also includes the so-called "quantum Boltzmann" distribution

$$n_i^{\text{QB}} = x_i \tag{16}$$

which, according to Greenberg (1990), obeys particles whose creation and annihilation operators satisfy the q-commutation relations (3) with $q=0$ [this distribution has the form (16) for any temperature and density rather than only in the Boltzmann limit $x_i \to 0$].

We turn now to the situation when it is more convenient to use a slightly different approach to the definition of the statistical distribution. Namely, let us consider the case when several quantum states correspond to every single-particle energy level, assuming that these states may have different properties with respect to the values their occupation numbers may take. A simple example is a supersymmetric particle which has both the bosonic and fermionic states for every energy level. We restrict ourselves here to the case when the total number and composition of states (the numbers of the bosonic, fermionic, and other states) are the same for all the single-particle energy levels. Denote the total number of states corresponding to every energy level (degeneracy of the energy levels) by g.

In the case at hand, it is convenient to attribute thermodynamic quantities to a given energy level rather than to a distinct state. Let the index I number single-particle energy levels and also label the thermodynamic quantities attributed to the level I. Split all the values of the index i numbering single-particle states into groups so that every group should contain the indices of the states belonging to the same energy level (we label these groups by the index I as well). Then the partition function Ξ_I and the thermodynamic potential Ω_I of the particles in all the states of energy ε_I can be written as

$$\Xi_I \in \prod_{i=I} \Xi_i, \qquad \Omega_I \in \sum_{i=I} \Omega_i \tag{17}$$

The thermodynamic potential of a gas of particles as a whole is

$$\Omega = \sum_I \Omega_I = \sum_{i=I} \Omega_i \tag{18}$$

Following the above considerations, we get the grand partition function Ξ_I as a function of $x_I \{\equiv \exp[\beta(\mu - \varepsilon_I)]\} : \Xi_I = \Xi_I(x_I)$. The statistical distribution $n_I = -(\partial\Omega_I/\partial\mu)_T$ (the average particle number of energy ε_I) is

$$n_I = \frac{x_I \Xi_I'(x_I)}{\Xi_I(x_I)} \tag{19}$$

Note that it is sometimes more convenient to discuss, instead of n_I, the average particle number of energy ε_I attributed to one state, $(1/g)n_I$.

In the case under consideration the condition of the existence of the Boltzmann limit is modified slightly. Indeed, since the Boltzmann distribution x_i is the average particle number in one quantum state but n_I is the average particle number in all g states of energy ε_I, then the condition of the existence of the Boltzmann limit takes the form $n_I \to g x_I$ for $x_I \to 0$ or

$$\frac{1}{g}\frac{\Xi_I'(x_I)}{\Xi_I(x_I)} \to 1 \qquad \text{as} \quad x_I \to 0 \tag{20}$$

To illustrate the above general consideration, we give some simple examples of the statistical distributions which are naturally written in the form (19). The first example is a gas of supersymmetric particles having the same number, to be denoted by $g/2$, of the bosonic and fermionic states for each energy value. In this case, the product in the first of equations (17) consists of $g/2$ multipliers $1+x_i$ and $g/2$ multipliers $(1-x_i)^{-1}$ corresponding to the fermionic and bosonic states, respectively. From (19), we get the natural result

$$\frac{1}{g} n_I = \frac{1}{2}\left(\frac{x_I}{1+x_I} + \frac{x_I}{1-x_I}\right) = \frac{1}{2}\left(\frac{1}{e^{\beta(\varepsilon_I - \mu)}+1} + \frac{1}{e^{\beta(\varepsilon_I-\mu)}-1}\right) \tag{21}$$

Generalizing (21), one can consider the situation when every single-particle energy level has g_{d_1} states such that each may be occupied by not more than d_1 particles, . . . , g_{d_l} states such that each may be occupied by not more than d_l particles ($g_{d_1}+g_{d_2}+\cdots+g_{d_l}=g$). Using in this case (9), with suitable d, in (17), we obtain from (19) the statistical distribution

$$\frac{1}{g} n_I = \frac{1}{g}\{g_{d_1}(n_I)_{d_1} + g_{d_2}(n_I)_{d_2} + \cdots + g_{d_l}(n_I)_{d_l}\} \tag{22}$$

where $(n_I)_{d_l}$ is given by (12), introduced by Balashova *et al.* (1989).

Note that the statistical distribution in the form (19) seems to be convenient for considering the statistical mechanics of paraparticles [when the latter reduce to particles with ordinary statistics; see, e.g., Govorkov (1973)], especially in the case when there is supersymmetry in a system (Hama *et al.*, 1991).

3. FINDING THE STATISTICAL DISTRIBUTION WITH THE AID OF MANY-PARTICLE PARTITION FUNCTIONS

In this section we are concerned with statistical distributions for which the partition function (13) allows an expansion in a series in integer powers of x_i:

$$\Xi_i(x_i) = 1 + x_i + \alpha_2(x_i)^2 + \alpha_3(x_i)^3 + \cdots \tag{23}$$

where $\alpha_2, \alpha_3, \ldots$ are coefficients. That the first term in the right side of (23) equals one means that the vacuum state is supposed to be nondegenerate. The coefficient of x_i in (23) has been chosen as equal to one in order to satisfy the condition of the existence of the Boltzmann limit (15).

The expansion (23) corresponds to a similar expansion of the statistical distribution (14)

$$n_i(x_i) = x_i + a_2(x_i)^2 + a_3(x_i)^3 + \cdots \tag{24}$$

The connection between the coefficients α_l and a_l ($l=2, 3, \ldots$) can be obtained with the aid of (14). Let us express, for instance, the coefficients α_l in terms of a_l. For this, we integrate (14) with the use of (24),

$$\begin{aligned} \Xi_i(x_i) &= \exp\left[\int_0^{x_i} \xi^{-1} n_i(\xi)\, d\xi\right] \\ &= \exp\left[\sum_{m=1}^{\infty} a_m(x_i)^m m^{-1}\right] \end{aligned} \tag{25}$$

[the constant of integration is fixed by the condition (15)]. Comparison with (23) yields

$$\alpha_l = ((x^l)) \exp\left[\sum_{m=1}^{\infty} a_m(x^m/m)\right] \tag{26}$$

where x^l in double parentheses denotes that one should take the coefficient at x^l in the expansion in a (formal) series in powers of x of the expression following $((x^l))$. The few first coefficients α_l are

$$\begin{aligned} &\alpha_2 = \tfrac{1}{2}(1 + a_2), \qquad \alpha_3 = \tfrac{1}{6}(1 + 3a_2 + 2a_3) \\ &\alpha_4 = \frac{1}{4!}(1 + 6a_2 + 8a_3 + 3a_2^2 + 6a_4), \ldots \end{aligned} \tag{27}$$

The expansion of the statistical distribution (24) is closely related, as we will see in Section 4, to the virial expansion of the equation of state for a quantum gas of noninteracting particles,

$$P = k_B T(n + B_2 n^2 + B_3 n^3 + \cdots) \tag{28}$$

where P is the pressure, n is equal to N/V (N/A) for three (two) dimensions, V (A) is the volume (the area) occupied by the gas, and N is the particle number. The virial coefficients $B_2, B_3, \ldots$ depend on the statistics of particles and thus contain certain information about it.

The virial coefficients can be expressed in terms of many-particle partition functions (see, e.g., Huang, 1963). Developing this idea, we find here the expressions for the coefficients a_l in terms of the many-particle partition functions of noninteracting particles.

For this purpose, we compare two expressions for the thermodynamic potential Ω for a system of N noninteracting particles. The former is known in theory of virial expansions (see, e.g., Osborn and Tsang, 1976). This is obtained from equation (6) (which obviously holds for closed system as well) after replacing $\mathcal{N}$ by N and taking into account that $\sum_k \exp(-\beta E_{Nk})$ is the N-particle partition function Z_N:

$$\Omega = -\beta^{-1} \ln \Xi = -\beta^{-1} \ln\left\{\sum_{N=0}^{\infty} \zeta^N Z_N\right\} \tag{29}$$

where $\zeta = e^{\beta\mu}$.

To find the second expression for Ω, we sum the equality $\Omega_i = -\beta^{-1} \ln \Xi_i$, with Ξ_i from (25), over i:

$$\Omega = \sum_i \Omega_i = -\beta^{-1} \sum_{l=1}^{\infty} a_l \zeta^l \frac{\sigma_l}{l} \tag{30}$$

where σ_l is defined by

$$\sigma_l = \sum_i e^{-l\beta\varepsilon_i} \tag{31}$$

By comparing the coefficients at the same powers of ζ in (30) and in the expansion in a series in ζ of (29), we obtain

$$a_l = ((\zeta^l)) \frac{l}{\sigma_l} \ln\left\{\sum_{N=0}^{\infty} Z_N \zeta^N\right\} \tag{32}$$

The few first coefficients a_l are

$$\begin{aligned} a_2 &= (2Z_2 - Z_1^2)/\sigma_2, \qquad a_3 = (3Z_3 - 3Z_2Z_1 + Z_1^3)/\sigma_3 \\ a_4 &= (4Z_4 - 4Z_3Z_1 - 2Z_2^2 + 4Z_2Z_1^2 - Z_1^4)/\sigma_4, \ldots \end{aligned} \tag{33}$$

So, if the many-particle partition functions for a system of noninteracting particles are known, the formulas (31)–(32) enable one to find the statistical distribution (24) [or, with the use of (26), the partition function (23)] in the form of a series in powers of x_i.

We will apply below the general considerations of this section to study concrete statistical distributions.

4. ON THE STATISTICAL DISTRIBUTIONS FOR ANYONS

Arovas *et al.* (1985) calculated the second virial coefficient for a system of noninteracting anyons with the quadratic dispersion law. We will translate here this result into the language of statistical distributions, having evaluated the coefficient a_2 in the expansion (24) for the statistical distribution for anyons.

We will thus propose that the statistical distributions for anyons, for arbitrary values of the parameter θ in (2), belong to the class of distributions allowing the expansion of the form (24). As a basis for this assumption, one can regard, first, the fact that it is true in special cases when the parameter θ in (2) takes values 0 and π and the statistical distributions coincide with the Bose and Fermi ones. In addition, for arbitrary θ, virial expansions of the form (28) hold for free anyons (Arovas *et al.*, 1985). Since the virial expansion (28) will be shown below to be a consequence of the expansion (24), it is natural to assume that an expansion of the form (24) holds for statistical distributions for anyons for arbitrary values of the parameter θ.

To find the connection between the coefficients B_2 and a_2, we transform the series (24) into a virial expansion of the kind of (28). For this, we first perform summation over i in (24), taking into account that $N=\sum_i n_i$ and restricting ourselves to the terms up to second order in x_i:

$$N=\sigma_1\zeta+a_2\sigma_2\zeta^2+\cdots \tag{34}$$

where σ_n is given by (31), $\zeta=e^{\beta\mu}$. Next, we will invert the series (34), having represented ζ in the form of a series in powers of N, restricting ourselves to the terms up to second order in N:

$$\zeta=\frac{1}{\sigma_1}N-a_2\frac{\sigma_2}{\sigma_1^3}N^2+\cdots \tag{35}$$

Finally, inserting (35) into (30) and taking into account that $\Omega=-PA$ in two dimensions, we get

$$PA=k_BT\left(N-\frac{1}{2}a_2\frac{\sigma_2}{\sigma_1^2}N^2+\cdots\right) \tag{36}$$

Thus, we have obtained that the virial expansion (36) [of the form (28)] is a consequence of the expansion (24). Comparing (36) with (28) yields a relation linking B_2 and a_2:

$$B_2=-\frac{1}{2}a_2\frac{\sigma_2}{\sigma_1^2}A \tag{37}$$

Arovas *et al.* (1985) calculated the second virial coefficient B_2 for the two-dimensional gas of noninteracting anyons with the quadratic dispersion law $\varepsilon_{\mathbf{p}}=p^2/2m$:

$$B_2=-\lambda_T^2(2\tilde{\Delta}^2-2\tilde{\Delta}+\tfrac{1}{4}) \tag{38}$$

where $\lambda_T=(2\pi\hbar^2/mk_BT)^{1/2}$ is the thermal length, $\tilde{\Delta}=\Delta-[\Delta]$ (brackets stand for integer part), $\Delta=\theta/2\pi$, and θ is the parameter appearing in (2) and taking any nonnegative real values here.

For the above anyon dispersion law, replacing the sum over i in (31) by an integral over momenta and evaluating the integral yields

$$\sigma_l = \int e^{-l\beta(p^2/2m)} \frac{d^2\mathbf{p}\, A}{(2\pi\hbar)^2} = \frac{A}{l\lambda_T^2} \tag{39}$$

Taking into account that $Z_1 = \sigma_1$ and inserting (38) and σ_1, σ_2 from (39) into (37), we obtain $a_2 = 8\tilde{\Delta}^2 - 8\tilde{\Delta} + 1$, so that the first terms of the expansion (24) for the statistical distribution for anyons are

$$n_i^{\text{anyon}} = x_i + (8\tilde{\Delta}^2 - 8\tilde{\Delta} + 1)x_i^2 + \cdots \tag{40}$$

Evaluating the coefficient α_2 with the help of (27), we obtain the first terms of the corresponding expansion (23) for the partition function:

$$\Xi_i^{\text{anyon}} = 1 + x_i + (2\tilde{\Delta} - 1)^2 x_i^2 + \cdots \tag{41}$$

For $\tilde{\Delta} = 0$ and $\tilde{\Delta} = \frac{1}{2}$, (40) and (41) agree with the appropriate expansions for bosons and fermions, respectively.

Notice that the coefficient a_2 in (40) (at x_i^2)—unlike the second virial coefficient B_2—does not depend on the anyon dispersion law, so the expansion (40) [and (41)] is valid for anyons with arbitrary dispersion law.

Notice an interesting fact. Let us consider a system of two anyons in a harmonic potential $V = \frac{1}{2}\omega^2(x^2 + y^2)$. The energy levels of the system are given by (Leinaas and Myrheim, 1977; Wilczek, 1982; Wu, 1984*b*)

$$E_2 = (2N + |L| + 2n + |l + 2\tilde{\Delta}| + 2)\hbar\omega \tag{42}$$

where N, n are nonnegative integers, L is an integer, and l is an even integer. The partition function evaluated with the aid of (42) is (Comtet *et al.*, 1989)

$$Z_2 = x^2(1 - x)^{-2}(1 - x^2)^{-2}(x^{2\tilde{\Delta}} + x^{2 - 2\tilde{\Delta}}) \tag{43}$$

where $x = e^{-\beta\hbar\omega}$.

The energy levels of a single anyon in a harmonic potential are simply those of a common two-dimensional oscillator

$$E_1 = (n_1 + n_2 + 1)\hbar\omega \tag{44}$$

where n_1, n_2 are nonnegative integers. This yields for σ_l of (31)

$$\sigma_l = x^l(1 - x^l)^{-2} \tag{45}$$

Let us consider the limit $\beta\hbar\omega \to 0$ (the limit of sufficiently high temperatures when the main contribution to the partition functions comes from the energy levels for which discreteness of the levels becomes inessential). Inserting Z_2 of (43) in (33), using (45) and the equality $Z_1 = \sigma_1$, we obtain that, at the leading order in $\beta\hbar\omega$, a_2 is equal to $8\tilde{\Delta}^2 - 8\tilde{\Delta} + 1$, which exactly agrees with (40) derived above by considering a system of noninteracting anyons.

This shows that the formulas of the previous section can be used for finding the successive terms of the expansion of the form (24) for the statistical distribution for anyons not only by considering a system of free anyons, but also in the case when the exact solutions of the many-anyon problem in a harmonic potential are known.

Very recently the virial coefficients for an anyon gas were estimated perturbatively near Bose and Fermi statistics (Sen, 1991; McCabe and Ouvry, 1991; Comtet *et al.*, 1991). It has been found that the virial coefficients B_N with $N \geq 3$ have no perturbative corrections near Bose statistics (when $\tilde{\Delta} \to 0$) at first order in $\tilde{\Delta}$, with respect to their bosonic values (Comtet *et al.*, 1991), and near Fermi statistics ($\delta = \frac{1}{2} - \tilde{\Delta} \to 0$) at first order in δ, with respect to their fermionic values (Sen, 1991). These results can be translated into the language of statistical distributions to give some conclusions concerning the statistical distributions for anyons. To illustrate this, we consider, as an example, the third virial coefficient B_3.

For anyons with a quadratic dispersion law, the above statements imply (Sen, 1991; Comtet *et al.*, 1991)

$$\begin{aligned} B_3(\tilde{\Delta}) &= [\tfrac{1}{36} + o(\tilde{\Delta})]\lambda_T^4, \qquad \tilde{\Delta} \to 0 \\ (B_3)_{\tilde{\Delta}=1/2-\delta} &= [\tfrac{1}{36} + o(\delta)]\lambda_T^4, \qquad \delta \to 0 \end{aligned} \tag{46}$$

Considerations similar to those that led to (37) for B_2 give for B_3

$$B_3 = \left(a_2^2 \frac{\sigma_2^2}{\sigma_1^4} - \frac{2}{3} a_3 \frac{\sigma_3}{\sigma_1^3}\right) A^2 \tag{47}$$

Combining this with (46), (39), and (33), we find for the coefficient a_3 in the expansion (24) for the statistical distribution for anyons

$$\begin{aligned} a_3(\tilde{\Delta}) &= 1 - 18\tilde{\Delta} + o(\tilde{\Delta}), \qquad \tilde{\Delta} \to 0 \\ (a_3)_{\tilde{\Delta}=1/2-\delta} &= 1 + o(\delta), \qquad \delta \to 0 \end{aligned} \tag{48}$$

and, according to (27), for the coefficient α_3 in the corresponding expansion for the partition function (23),

$$\begin{aligned} \alpha_3(\tilde{\Delta}) &= 1 - 10\tilde{\Delta} + o(\tilde{\Delta}), \qquad \tilde{\Delta} \to 0 \\ (\alpha_3)_{\tilde{\Delta}=1/2-\delta} &= o(\delta), \qquad \delta \to 0 \end{aligned} \tag{49}$$

5. STATISTICAL DISTRIBUTIONS CORRESPONDING TO THE q-COMMUTATION RELATIONS AND TO SOME OF THEIR GENERALIZATIONS

5.1. Statistical Distributions for Quons

Here we find the statistical distributions for quons, particles whose creation and annihilation operators satisfy the q-commutation relations (3) with q in the interval $-1 \leq q \leq 1$. We will suppose that all these belong to the class of distributions allowing an expansion of the form (24). This assumption seems to be natural because it is valid for the Bose and Fermi distributions corresponding to the relations (3) with q at the ends of the interval $-1 \leq q \leq 1$. We will see below the consistency of this assumption.

First, as for anyons in the previous section, we evaluate the coefficient a_2 in the expansion (24) for the statistical distribution for quons. To estimate the two-particle partition function Z_2 on the right side of the first equality in (33) in the case involved, we use the following arguments.

There are only two irreducible representations of the permutation group S_2, symmetric and antisymmetric, corresponding to the Young diagrams

$$\square\square \quad \text{and} \quad \begin{smallmatrix}\square\\\square\end{smallmatrix}$$

Mohapatra (1990) found that any two-quon state is the mixture of the symmetric and antisymmetric states with the probabilities

$$w_{\square\square} = \tfrac{1}{2}(1+q), \qquad w_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \tfrac{1}{2}(1-q) \tag{50}$$

(for $q=1$, the symmetric state occurs with probability one; for $q=-1$, the antisymmetric state does). Note that this agrees with the approach of Hartle and Taylor (1969) in thinking of a general N-particle state as a mixture of states with symmetries each corresponding to the symmetry of some irreducible representation of the permutation group S_N.

In accordance with this, we write the two-particle partition function of noninteracting quons Z_2 in the form

$$Z_2 = w_{\square\square} Z_{\square\square} + w_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} Z_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \tag{51}$$

where

$$Z_{\square\square} \quad \text{and} \quad Z_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$$

are the partition functions containing contributions from the two-particle states having symmetries

$$\square\square \quad \text{and} \quad \begin{smallmatrix}\square\\\square\end{smallmatrix}$$

that is, the partition functions of noninteracting bosons and fermions, respectively. One can represent

$$Z_{\square\square} \quad \text{and} \quad Z_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$$

as

$$Z_{\square\square} = Z_2^{\mathrm{B}} = Z_{(2)} + Z_{(1,1)}, \qquad Z_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = Z_2^{\mathrm{F}} = Z_{(1,1)} \tag{52}$$

where the partition function $Z_{(2)}$ ($Z_{(1,1)}$) includes the contributions from those states of a system of two identical particles which correspond to the two particles in the same quantum state (in distinct states). Let a set of the energy values of one particle be $\{\varepsilon_i\}$ with i taking values in some index set. Then, one can write for $Z_{(2)}$ and $Z_{(1,1)}$,

$$
\begin{aligned}
Z_{(2)} &= \sum_i e^{-2\beta\varepsilon_i} = \sigma_2 \\
Z_{(1,1)} &= \tfrac{1}{2} \sum_i e^{-\beta\varepsilon_i} \sum_{j(\neq i)} e^{-\beta\varepsilon_j} \\
&= \tfrac{1}{2} \sum_i e^{-\beta\varepsilon_i} \left(\sum_j e^{-\beta\varepsilon_j} - e^{-\beta\varepsilon_i} \right) = \tfrac{1}{2}(\sigma_1^2 - \sigma_2)
\end{aligned}
\tag{53}
$$

with σ_n given by (31). We get from (52) and (53)

$$
Z_{\square\square} = Z_2^{\mathrm{B}} = \tfrac{1}{2}(\sigma_1^2 + \sigma_3), \qquad Z_{\genfrac{}{}{0pt}{}{\square}{\square}} = Z_2^{\mathrm{F}} = \tfrac{1}{2}(\sigma_1^2 - \sigma_2) \tag{54}
$$

Inserting (54) into (51) and evaluating the coefficient a_2 of (33) (recall that $Z_1 = \sigma_1$) with the aid of the resulting expression for Z_2 yields

$$
a_2 = q \tag{55}
$$

Thus, the first terms of the expansion (24) for the statistical distribution for quons are

$$
n_i^{\mathrm{quon}} = x_i + q x_i^2 + \cdots \tag{56}
$$

From (27), we obtain $\alpha_2 = (1+q)/2$, so that the corresponding expansion for the partition function (23) is

$$
\Xi_i^{\mathrm{quon}} = 1 + x_i + \tfrac{1}{2}(1+q)x_i^2 + \cdots \tag{57}
$$

For $q = 1$ and $q = -1$, (56) and (57) coincide with the first terms of suitable expansions for bosons and fermions. Furthermore, note that these agree with Greenberg's (1990) statement that for $q = 0$, the statistical distribution for quons is the "quantum Boltzmann" one (16).

We now find the statistical distribution for quons in full. For this purpose, we use the representation—given by Mohapatra (1990)—of the q-commutation relations (3) for the particles with a single quantum state. In the latter case, on omitting indices i, j, (3) becomes

$$
aa^+ - qa^+a = 1 \tag{58}
$$

One can introduce an infinite set of orthonormal states $\{|N\rangle\}$, $N = 0, 1, 2, \ldots$, on which the operators a^+, a satisfying (58) act as the raising and lowering operators:

$$a^{+}|N\rangle=\lambda_{N+1}|N+1\rangle, \qquad N=1,2,\ldots$$
$$a|N\rangle=\lambda_{N}|N-1\rangle, \qquad N=0,1,2,\ldots \tag{59}$$

The recursion relation is obtained for the coefficients λ_N:

$$\lambda_{N+1}^2=1+q\lambda_N^2 \tag{60}$$

which, combined with the natural condition $\lambda_0=0$ being the consequence of the vacuum condition $a|0\rangle=0$ and (59), yields

$$\lambda_N^2=1+q+\cdots+q^{N-1} \tag{61}$$

Furthermore, the equality follows from (59),

$$a^{+}a|N\rangle=\lambda_N^2|N\rangle \tag{62}$$

The formulas (59)–(62) obtained by Mohapatra (1990) give the representation of the q-commutation relations (58) on the space spanned by the vectors $\{|N\rangle\}$, $N=0, 1, 2, \ldots$, to which we will refer as the representation in the *bosonic Fock space.*

We consider the Hamiltonian (see, e.g., Greenberg, 1990)

$$H=\varepsilon\hat{N} \tag{63}$$

where ε is the particle energy, and $\hat{N}$ is the particle number operator having by definition eigenvalue N on the state $|N\rangle$:

$$\hat{N}|N\rangle=N|N\rangle \tag{64}$$

For $q=1$, equations (59)–(62) give the representation of the commutation relations for Bose operators. In this case, the particle number operator $\hat{N}$ is $a^{+}a$. Computing the statistical distribution (the average particle number in the state of energy ε) in the usual way,

$$n=\frac{\mathrm{Tr}\{\hat{N}\exp[\beta(\mu\hat{N}-H)]\}}{\mathrm{Tr}\exp[\beta(\mu\hat{N}-H)]} \tag{65}$$

where the trace is performed over all the states $\{|N\rangle\}$, we get

$$n=\frac{\sum_{N=0}^{\infty}Nx^N}{\sum_{N=0}^{\infty}x^N}=\frac{x}{1-x}=\frac{1}{e^{\beta(\varepsilon-\mu)}-1} \tag{66}$$

with $x=e^{\beta(\mu-\varepsilon)}$, that is, the Bose distribution for the particles having a single state of energy ε.

We now ask the following question. Can the statistical distributions corresponding to the commutation relations (3) for q other than one be obtained by using only the above representation of the q-commutation relations in the bosonic Fock space? In particular, can the Fermi distribution be obtained for $q=-1$ in such a way?

We propose here an ansatz according to which the statistical distributions corresponding to the commutation relation (3) for arbitrary q in $-1\leq q\leq 1$ can be evaluated with the help of the following formula:

$$n=\langle a^{+}a\rangle=\frac{\mathrm{Tr}\{a^{+}a\exp[\beta(\mu\hat{N}-H)]\}}{\mathrm{Tr}\exp[\beta(\mu\hat{N}-H)]} \tag{67}$$

Taking into account (63)–(64), we obtain

$$\mathrm{Tr}\exp[b(\mu\hat{N}-H)]=1+x+x^{2}+\cdots=(1-x)^{-1}$$

Also using (62), we can rewrite equation (67) as

$$n=(1-x)\sum_{N=0}^{\infty}\lambda_{N}^{2}x^{N}\equiv(1-x)D(x) \tag{68}$$

For $q=1$, the ansatz (67) recovers the Bose distribution since in this case the particle number operator $\hat{N}$ is $a^{+}a$ and therefore (67) reduces to (65).

Consider the case $q=-1$. In this case, λ_{N}^{2} of (61) is equal to zero or one, depending on whether N is even or odd. Then, according to (68), we obtain

$$n=(1-x)(x+x^{3}+x^{5}+\cdots)=\frac{x}{1+x}=\frac{1}{e^{\beta(\varepsilon-\mu)}+1} \tag{69}$$

which is the Fermi distribution. Thus, the ansatz proposed using only the representation of the q-commutation relations in the bosonic Fock space does nevertheless recover for $q=-1$ the correct result, the Fermi distribution. This suggests that the ansatz also can be successful in evaluating the statistical distributions for different values of q in the interval $-1\leq q\leq 1$.

Let us consider now arbitrary q in $-1\leq q\leq 1$. For evaluating $D(x)$ in (68) in this case, we multiply both the sides of equation (60) by x^{N+1} and perform summation over N nonnegative integers in the resulting equality:

$$\sum_{N=1}^{\infty}\lambda_{N}^{2}x^{N}=\sum_{N=1}^{\infty}x^{N}+q\sum_{N=1}^{\infty}\lambda_{N-1}^{2}x^{N} \tag{70}$$

[in (70) the change of the summation variable $N\to N-1$ is made]. Since $\lambda_{0}=0$, the second term on the right side of (70) can be rewritten in the form $qx\sum_{N=2}^{\infty}\lambda_{N-1}^{2}x^{N-1}$, showing the proportionality of it to $D(x)$. Thus

$$D(x)=\frac{x}{(1-x)(1-qx)} \tag{71}$$

Inserting (71) into (68), we obtain finally the following simple expression for the statistical distribution for quons:

$$n^{\text{quon}} = \frac{x}{1-qx} = \frac{1}{e^{\beta(\varepsilon-\mu)}-q} \tag{72}$$

which interpolates between the Bose ($q=1$) and Fermi ($q=-1$) distributions.

Let us expand (72) in a series in powers of x,

$$n^{\text{quon}} = x(1+qx+q^2x^2+\cdots) \tag{73}$$

The coefficient at x^2 in (73) coincides with what has been found above in a different way [see (56)]. This supports the ansatz suggested.

Furthermore, for $q=0$, (72) yields

$$(n^{\text{quon}})_{q=0} = x = e^{\beta(\mu-\varepsilon)} \tag{74}$$

agreeing exactly with Greenberg's (1990) statement that for $q=0$ the q-commutation relations should correspond to the "quantum Boltzmann" distribution (16). This also supports the ansatz (67).

These facts support the consistency of the ansatz proposed. Below, the additional arguments supporting the ansatz (67) will be presented.

By using the first equality in (25), one can find the partition function corresponding to the statistical distribution (72):

$$\Xi^{\text{quon}}(x) = (1-qx)^{-1/q} \tag{75}$$

Expansion in powers of x yields

$$\Xi^{\text{quon}}(x) = \sum_{s=0}^{\infty} \frac{(l!)_q}{l!} x^l \tag{76}$$

with

$$(l!)_q \equiv (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{l-1})$$

In particular, for $q=0$, one obtains $\Xi(x)=e^x$.

The ansatz (67)–(68) can be used for evaluating the statistical distribution corresponding to different commutation relations for which the representation in the bosonic Fock space is given. We consider here the commutation relations (Mohapatra, 1990)

$$aa^+ - qa^+a = q^{-2\hat{N}} \tag{77}$$

where $\hat{N}$ is the particle number operator, which have been constructed by analogy with those arising in theory of quantum groups, $aa^+ - qa^+a = q^{-\hat{N}}$ (Biedanharn, 1989; Macfarlane, 1989; Sun and Fu, 1989). Like (58), the commutation relations (77) interpolate between those for Bose ($q=1$) and Fermi ($q=-1$) operators. Mohapatra (1990) found that for the case (77), the eigenvalues of the operator a^+a on the above-introduced states $|N\rangle$ are

$$\lambda_N^2=q^{N-1}(1-q^{-3N})/(1-q^{-3}) \tag{78}$$

With these λ_N, the evaluation of the statistical distribution with the aid of (68) (computing the sum over N reduces to summing a geometric progression) results in

$$n=\frac{x(1-x)}{(1-qx)(1-x/q^2)} \tag{79}$$

interpolating between the Bose ($q=1$) and Fermi ($q=-1$) distributions. That the Fermi distribution is obtained for $q=-1$ appears to indicate the applicability of the ansatz (67) to the commutation relations (77). Note that (79) is inconsistent for $q=0$ since $n\to 0$ when $q\to 0$.

5.2. Generalizations of the q-Commutation Relations

Here we propose some generalizations of the q-commutation relation for single-state systems (58) and outline a way of evaluating the corresponding statistical distributions.

On rewriting the commutation relation (58) as

$$aa^+=1+qa^+a \tag{80}$$

we will consider successive generalizations of the relations (80), adding to the right side of (80) a normal product of the operators a, a^+, quadratic in a, a^+,

$$aa^+=1+q_1a^+a+q_2a^+a^+aa \tag{81}$$

a sum of normal products of the operators a, a^+ up to degree d,

$$aa^+=\sum_{l=1}^{d} q_l(a^+)^l a^l \tag{82}$$

and, finally, an infinite series of normal products of the operators a, a^+,

$$aa^+=\sum_{l=1}^{\infty} q_l(a^+)^l a^l \tag{83}$$

In (81)–(83), $q_1, q_2, \ldots$ are coefficients.

We will discuss a representation of the commutation relations (81)–(83) in the bosonic Fock space, similar to Mohapatra's (1990) consideration of the q-commutation relations. Namely, we suppose the equality (59) [and hence, (62)] to hold. We will apply the above ansatz and consider what it leads to.

First, we treat the commutation relation (81). As above, $\lambda_0 = 0$ as a consequence of the vacuum condition $a|0\rangle = 0$ and (59). Next, as for the relation (58), the application of both sides of (81) to the state $|0\rangle$ yields, in virtue of (59), $\lambda_1^2 = 1$. Let, as above, $\lambda_1 = 1$. The application of (81) to the states $|N\rangle$ with $N \geq 1$ results in the recursion relation

$$\lambda_{N+1}^2 = 1 + q_1\lambda_N^2 + q_2\lambda_{N-1}^2\lambda_N^2 \tag{84}$$

[reducing to (60) for $q_1 = q$, $q_2 = 0$].

When $q_1 = 1$, $q_2 = -\frac{3}{2}$, the commutation relation (81) reduces to that for particles obeying 2-statistics (Okayama, 1952; Kuryshkin, 1988):

$$aa^+ = 1 + a^+a - \tfrac{3}{2}a^+a^+aa \tag{85}$$

Since $\lambda_0 = 0$, $\lambda_1 = 1$, we obtain from (84) in this case that $\lambda_{3m} = 0$, $\lambda_{3m+1} = 1$, and $\lambda_{3m+2} = 2$ for m nonnegative integers. Then the statistical distribution obtained with the aid of (68),

$$n = \frac{x(1+2x)}{1+x+x^2} \tag{86}$$

coincides with that for particles obeying 2-statistics [(12) with $d = 2$]. Thus, the ansatz (67)–(68) leads to the correct statistical distribution in the special case when the commutation relation (81) reduces to (85).

Similarly, for the commutation relation (82), one can get the recursion relation generalizing (60), (84):

$$\lambda_{N+1}^2 = 1 + q_1\lambda_N^2 + q_2\lambda_{N-1}^2\lambda_N^2 + \cdots + q_d\lambda_{N-d+1}^2\lambda_{N-d+2}^2\cdots\lambda_N^2 \tag{87}$$

where all λ's with negative subscripts should be regarded as vanishing. In the special case when $q_1 = 1$, $q_2 = \cdots = q_{d-1} = 0$, and $q_d = -(d+1)/d!$, the commutation relation (82) reduces to that for particles obeying d-statistics (Kuryshkin, 1988):

$$aa^+ = 1 + a^+a - \frac{d+1}{d!}(a^+)^d(a)^d \tag{88}$$

In this case, the ansatz (68), together with the relations (87), leads to the statistical distribution for particles obeying d-statistics (12).

Thus, for special values of $q_1, \ldots, q_d$, when the commutation relations (81)–(82) correspond to d-statistics with different d, the ansatz (67)–(68) leads to the correct results for statistical distributions. This suggests that the ansatz can be successful in finding the statistical distributions corresponding to the commutation relations (81)–(82) for arbitrary values of $q_1, \ldots, q_d$

[and also to the commutation relation (83) as the limiting case $d \to \infty$ of (82)].

Let us turn now to the commutation relation of the most general form (83). As always above, we have $\lambda_0 = 0$, and λ_N with $N = 1, 2, \ldots$ are found consecutively from the equalities which are obtained by application of both sides of equation (83) to the states $|N\rangle$ with $N = 1, 2, \ldots$. The first λ_N are

$$\lambda_1^2 = 1, \qquad \lambda_2^2 = 1 + q, \qquad \lambda_3^2 = 1 + (1 + q_1)(q_1 + q_2), \ldots \tag{89}$$

Using (68), one can evaluate the first terms in the expansion (24) for the statistical distribution:

$$n = x + q_1 x^2 + (q_1^2 + q_1 q_2 + q_2) x^3 + \cdots \tag{90}$$

Since, in virtue of (62), the eigenvalues of a^+a on the states $|N\rangle$ are equal to λ_N^2, only λ_N with $N \leq l$ contribute to the coefficient at x^l in (90). Therefore, computing consecutively the quantities $\lambda_1, \lambda_2, \ldots$ of (89), one can evaluate consecutively the coefficients in the series (24) and obtain finally the statistical distribution in the form of an expansion in an infinite series in powers of x.

6. A UNIFIED DESCRIPTION OF A NONEQUILIBRIUM QUANTUM GAS

So far, we have dealt with gases in a state of thermodynamic equilibrium. This section is devoted to a unified description of a nonequilibrium gas. We will be concerned with the nonequilibrium entropy, one of the most important characteristics of a nonequilibrium gas, and also consider the collision integral. These have different forms for bosons and fermions. We will obtain unified expressions for the nonequilibrium gas entropy and for the collision integral which are valid for a wide class of statistics.

6.1. The Entropy of a Nonequilibrium Quantum Gas

Let us consider first the expression for the entropy of an equilibrium gas obeying the statistical distribution (14). It is derived with the use of the known thermodynamic identity $S = -(\partial\Omega/\partial T)_\mu$ applied to (18), where Ω_i is given by the second of equations (11), and Ξ_i in (11) is obtained by integrating equation (14):

$$S = \sum_i S_i = \sum_i \left\{ \ln \Xi_i(x_i) - \frac{\Xi_i'(x_i)}{\Xi_i(x_i)} x_i \ln x_i \right\} \tag{91}$$

Let $x_i(n_i)$ be x_i expressed in terms of n_i with the aid of equation (14) treated as an equation for x_i. We substitute $x_i(n_i)$ into (91) instead of x_i:

$$S=\sum_i S_i=\sum_i \{\ln \Xi_i(x_i)-n_i \ln x_i\}_{x_i=x_i(n_i)} \tag{92}$$

Let us notice now that the expression for the entropy S_i in terms of the distribution function n_i should have the same form whether the distribution is equilibrium or nonequilibrium. This follows from the usual way of determining the nonequilibrium gas entropy by summing the statistical weights corresponding to the sets of states with near energy values [see, e.g., Landau and Lifshitz (1980) and also the Appendix; we will refer to this as the *Boltzmann approach*], which uses in no way the fact whether gas is in equilibrium or not. Therefore, (92) is the entropy of a nonequilibrium gas obeying the distribution (14) in equilibrium.

The expression (92) can be obtained in an alternative way. Namely, if one tries to find the most general expression for the entropy S_i in terms of n_i, which, in varying with respect to n_i, provided the total particle number and the total gas energy are constants, would lead to the equilibrium statistical distribution (14) as the extremum condition, then (92) is easy to obtain.

In special cases, (92) reduces to the expressions for the entropy of nonequilibrium Bose and Fermi gases. It would be desirable to make sure explicitly that (92) agrees with what is obtained in the Boltzmann approach for statistics other than Bose–Einstein and Fermi–Dirac ones. One such example is given in the Appendix, where the entropy of a nonequilibrium gas of particles obeying 2-statistics is calculated in the Boltzmann approach and is shown to coincide with what is obtained from the general formula (92).

6.2. A General Form for the Collision Integral

In treating the collision integral we will consider, for definiteness, the case when two particles obeying the same statistics take part in an elementary collision act. In this case, the Bose–Einstein and Fermi–Dirac statistics, the collision integral reads (see, e.g., Ahiezer and Peletminskiy, 1977)

$$I(n_{\mathbf{p}})=\sum_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4} \delta_{\mathbf{p}.\mathbf{p}_4} w(\mathrm{p1p2},\ \mathrm{p3p4})\,\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)$$
$$\times \{n_{\mathbf{p}_1}n_{\mathbf{p}_2}(1\pm n_{\mathbf{p}_3})(1\pm n_{\mathbf{p}_4})-n_{\mathbf{p}_3}n_{\mathbf{p}_4}(1\pm n_{\mathbf{p}_1})(1\pm n_{\mathbf{p}_2})\} \tag{93}$$

where the upper signs refer to bosons and the lower signs to fermions. In (93), $n_{\mathbf{p}}$ is the particle distribution function, w is determined by the particle interaction potential, and the two terms in braces correspond to the process $\mathbf{p}_1+\mathbf{p}_2\rightarrow \mathbf{p}_3+\mathbf{p}_4$ and to the inverse one. We will only discuss here the first process and, properly, the first term in braces in (93) (the inverse process can be treated similarly); the corresponding part of the expression under the summation sign in (93) is the probability of the collision $\mathbf{p}_1+\mathbf{p}_2 \to \mathbf{p}_3+\mathbf{p}_4$.

When we speak about the form of the collision integral, we refer to the dependence of the collision probability on the particle distribution functions in the initial ($n_{\mathbf{p}_1}, n_{\mathbf{p}_2}$) and the final ($n_{\mathbf{p}_3}, n_{\mathbf{p}_4}$) states. The dependence on $n_{\mathbf{p}_1}, n_{\mathbf{p}_2}$ both for bosons and for fermions consists simply in proportionality to the product $n_{\mathbf{p}_1}n_{\mathbf{p}_2}$, or equivalently, to the product of the numbers of the colliding particles. The statistics of the particles is displayed through the dependence of the collision probability on the particle distribution functions in the final states, which has the form $(1+n_{\mathbf{p}_3})(1+n_{\mathbf{p}_4})$ and $(1-n_{\mathbf{p}_3})(1-n_{\mathbf{p}_4})$ for bosons and fermions, respectively.

Let us observe now that for bosons, the latter dependence can be obtained by using the known Einstein relations for the emission and absorption probabilities. Indeed, let us write these connecting the probability of the absorption of a photon in state i by a system of atoms $w_{\text{ab}}(i)$ with that of the emission of a photon in the same state $w_{\text{em}}(i)$ as follows:

$$\frac{w_{\text{ab}}(i)}{w_{\text{em}}(i)}=\frac{n_i}{1+n_i} \tag{94}$$

where n_i is the photon distribution function.

The relation (94) can be treated as connecting the probability of a photon leaving the ith state, $w_{\text{ab}}(i)$, with that of a photon coming into the same state, $w_{\text{em}}(i)$. In such a formulation, the relation (94) has a wider applicability domain. Namely, for any statistical processes with bosons taking part, these connect the probability of a boson coming into the ith state, $w_{\text{em}}(i)$, with that of a boson leaving this state, $w_{\text{ab}}(i)$ (n_i is the statistical distribution for bosons now), valid not only for equilibrium processes but also for nonequilibrium ones.

As applied to the collision integral, this implies the following. Since the probability of leaving state i has been shown above to be proportional to the particle number in this state n_i, then the probability of coming into state i, according to (94), will be proportional to $1+n_i$. Thus, we come to the above form for the collision integral for bosons (93).

Exploiting these observations, we will give below a generalization of the Einstein relation (94) to find on this basis a general form for the collision integral applicable to a wide class of statistics.

6.2.1. Generalization of the Einstein Relations

To generalize (94), we consider a generalization of Einstein's (1916) picture of the absorption and emission of photons by atoms. Namely, let us imagine a system consisting of particles which obey arbitrary statistics belonging to the class (14) (we will call them "particles" simply for short) and some hypothetical objects, with discrete energy levels (to be referred as "atoms"), able to emit and absorb the "particles." Thermodynamic equilibrium in such a system is assumed to be reached as a result of the "atoms" exchanging "particles."

Einstein (1916), using the relation (94), recovered the Planck distribution for photons. We will act in an opposite way: starting with the absorbed and emitted particles obeying the statistical distribution (14), we will derive for these the relations analogous to (94).

Following Einstein (1916), we regard the distribution of "atoms" over their energy levels as the Boltzmann one. Then the ratio of the number of "atoms" in the states of energy ε_m to that of energy ε_l is

$$N_m/N_l = (g_m/g_l)\, e^{\beta(\varepsilon_l - \varepsilon_m)} \tag{95}$$

where g_m, g_l are the degeneracies of the energy levels. Let $h\omega_{lm} = \varepsilon_l - \varepsilon_m > 0$. The "atoms" pass from the lth level into the mth level and vice versa, emitting and absorbing "particles" of energy $h\omega_{lm}$. In equilibrium, this number should be equal to the number of "atoms" passing from the lth level to the mth level, $N_l g_m w_{\text{em}}$, where w_{em} is the emission probability. This equality combined with (95) yields

$$w_{\text{ab}}/w_{\text{em}} = e^{-\beta h\omega_{lm}} \tag{96}$$

Replacing in (96) the indices lm by single index i and the photon energy $h\omega_{lm}$ by ε_i, and then expressing $\exp(-\beta\varepsilon_i)$ in terms of the particle distribution function n_i, with the aid of equation (14), we get the generalization of the Einstein relation (94) to the case of particles obeying any statistics belonging to the class (14):

$$w_{\text{ab}}(i)/w_{\text{em}}(i) = x_i(n_i) \tag{97}$$

where $x_i(n_i)$ is again the solution of equation (14) treated as an equation for x_i.

6.2.2. *A General Form for the Collision Integral*

We use now the relation (97) in order to generalize the collision integral (93) to particles obeying in equilibrium the statistical distribution (14).

We believe, first, that, like (94), the relation (97) holds not only for equilibrium processes, but also for nonequilibrium ones. In addition, we consider, as above, the probability of the collision $\mathbf{p}_1 + \mathbf{p}_2 \to \mathbf{p}_3 + \mathbf{p}_4$ to be proportional to $n_{\mathbf{p}_1} n_{\mathbf{p}_2}$, that is, to the product of the numbers of colliding particles. The latter means that the probability of a particle leaving state i is proportional to n_i with some factor independent of n_i. In accordance with the above interpretation of the Einstein relation extended to their generalization (97), the probability of a particle coming into state i is then proportional to

$$f(n_i)=\frac{n_i}{x(n_i)} \tag{98}$$

with the same factor. Thus, the dependence of the probability of the process $\mathbf{p}_1+\mathbf{p}_2\to\mathbf{p}_3+\mathbf{p}_4$ on the particle distribution functions in the final states $n_{\mathbf{p}_3}$ and $n_{\mathbf{p}_4}$ shows up in the factor $f(n_i)$ for every one of these states. Hence the collision integral for particles obeying in equilibrium the statistical distribution (14) can be written in the form [compare with (93)]

$$\begin{aligned} I(n_{\mathbf{p}}) = \sum_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4} & \delta_{\mathbf{p},\mathbf{p}_4} w(\mathbf{p}_1\mathbf{p}_2, \mathbf{p}_3\mathbf{p}_4)\, \delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4) \\ & \times \{n_{\mathbf{p}_1}n_{\mathbf{p}_2}f(n_{\mathbf{p}_3})f(n_{\mathbf{p}_4}) - n_{\mathbf{p}_3}n_{\mathbf{p}_4}f(n_{\mathbf{p}_1})f(n_{\mathbf{p}_2})\} \end{aligned} \tag{99}$$

with $f(n_i)$ given by (98).

We note, first, that the expression (99) leads to the correct result for fermions, for which $x_i(n_i)=n_i/(1-n_i)$ and, hence, $f(n_i)=1-n_i$. Next, it satisfies the natural demand concerning a collision integral. Namely, it vanishes for the equilibrium distribution function. Furthermore, by using the expression (92) for a nonequilibrium entropy, Boltzmann's H-theorem can be derived in the case at hand, which indicates the consistency of the expression (99) for the collision integral with the expression for a nonequilibrium entropy (92). All these facts point to the self-consistency of the result (99). This suggests that, in spite of derivation being somewhat heuristic, (99) is the correct expression for the collision integral for particles obeying any statistics belonging to the class (14).

7. CONCLUDING REMARKS

In the preceding sections we have dealt with the class of statistics appearing in the case when the condition is fulfilled that the quantum states are filled by particles independent of each other. Note that the more general situation is logically possible when, for some given set of quantum states, filling any state depends on the particle numbers in the other states of the set. In this case, it is necessary to deal with the partition function for the particles in all the states of the set because this cannot be factorized now (cannot be presented as the product of the partition functions, each corresponding to the particles being in one quantum state). The latter may take place for parastatistics (Bhattacharyya *et al.*, 1989*a*,*b*). Note that this more involved situation permits a general treatment similar to what was presented in the preceding sections. The appropriate work is in progress.

A further remark concerns the following question: what does giving the statistics of particles mean in the approach dealing with the symmetries of many-particle state vectors? We would like to note here the following. In order to give the statistics of particles, it is necessary to give the symmetries of the N-particle state vectors for all N, which can be pictured as giving an infinite sequence of symmetries

$$Symm(2) \to Symm(3) \to \cdots \to Symm(N) \to \cdots \tag{100}$$

where $Symm(N)$ denotes conditionally the symmetry of the N-particle state vector. Indeed, if the latter is known for a given N, it permits in principle the evaluation of the partition function of N noninteracting particles Z_N. Giving the whole sequence (100) enables one to evaluate all the many-particle partition functions and, according to the results of Section 3, to find the statistical distribution. It is natural to regard the statistics as being given when the statistical distribution is known. Therefore, the problem of classification of statistics in the approach dealing with the symmetries of many-particle state vectors can be formulated as the problem of classification of the allowed symmetry sequences (100).

We note in conclusion that the complete theory will unify all three of the above approaches to a generalization of statistics, that is, in addition to the correspondence between allowed symmetry sequences (100) and statistical distributions, this should also show, for every allowed symmetry sequence (100), the commutation relations for creation and annihilation operators for particles whose many-particle state vectors possess the proper symmetries.

APPENDIX. THE ENTROPY OF A NONEQUILIBRIUM GAS OBEYING 2-STATISTICS: THE GENERAL FORMULA (92) AND THE BOLTZMANN APPROACH

Here we will derive, using the Boltzmann approach, the expression for the entropy of a nonequilibrium gas obeying 2-statistics and compare this with the expression obtained from the general formula (92).

We turn first to the formula (92). For 2-statistics, according to (9), $\Xi_i(x_i) = 1 + x_i + x_i^2$. Inserting this into (14), we obtain an equation linking n_i and x_i, quadratic with respect to x_i. A positive root of this equation is $x_i(n_i) = 2n_i \{C(n_i) + 1 - n_i\}^{-1}$, where

$$C(n_i) = (1 + 6n_i - 3n_i^2)^{1/2} \tag{A.1}$$

Thus, according to (92), the entropy of a nonequilibrium gas obeying 2-statistics is

$$S=\sum_i \left[-\ln\left(\frac{7-3n_i+C(n_i)}{6}\right)+n_i \ln\left(\frac{C(n_i)+1-n_i}{2n_i}\right)\right] \tag{A.2}$$

Let us turn now to the Boltzmann approach. In this approach (see, e.g., Landau and Lifshitz, 1980) all the single-particle quantum states are split into groups of states with near energy values. We label the state groups by the index j. Let the number of states in the jth group be G_j, and the number of particles in the states belonging to the jth group be N_j. Then the nonequilibrium gas entropy is defined as

$$S=\sum_j S_j=\sum_j \ln(\Delta\Gamma_j) \tag{A.3}$$

where $\Delta\Gamma_j$ is the statistical weight for the jth group of particles (the number of ways of placing N_j particles over G_j states).

We estimate (A.3) for a gas obeying 2-statistics. Let N_{1j} be the number of states occupied in some placing only by one particle. Given N_{1j}, the number of ways of placing N_j particles over G_j states is equal to the number of ways by which $(N_j+N_{1j})/2$ occupied states can be chosen from the total number of states G,

$$\binom{G_j}{(N_j+N_{1j})/2}$$

multiplied by the number of ways by which N_{1j} states can be chosen from the number of occupied states,

$$\binom{(N_j+N_{1j})/2}{N_{1j}}$$

where

$$\binom{G}{N}\equiv\frac{G!}{N!(G-N)!}$$

is a binomial coefficient. Then

$$\Delta\Gamma_j=\sum_{N_{1j}}\binom{G_j}{(N_j+N_{1j})/2}\binom{(N_j+N_{1j})/2}{N_{1j}} \tag{A.4}$$

The values over which the summation in (A.4) is performed are determined by the condition that the number of occupied states should not exceed G_j, which gives $(N_j+N_{1j})/2\leq G_j$ or $N_{1j}\leq 2G_j-N_j$, that is,

$$N_{1j}=0, 1, \ldots, 2G_j-N_j$$

Since N_j, N_{1j}, G_j are macroscopically large numbers, one can use Stirling's formula $N! \approx \exp(N \ln N)$ for $N \gg 1$ in (A.4). Introducing the notations $n_j = N_j/G_j$, $n_{1j} = N_{1j}/G_j$ and replacing the resulting sum over n_{1j} in (A.4) by an integral, then, since G_j^{-1} is extremely small, we obtain

$$\Delta\Gamma_j = G_j \int_0^{2-n_j} \exp\{-G_j f(z)\}\, dz \tag{A.5}$$

where

$$f(z) = \left(1 - \frac{n_j + z}{2}\right) \ln\left(1 - \frac{n_j + z}{2}\right) + z \ln z + \left(\frac{n_j - z}{2}\right) \ln\left(\frac{n_j - z}{2}\right) \tag{A.6}$$

We estimate the leading term in the asymptotic expansion of the integral (A.5) for $G_j \gg 1$ by the Laplace method (see, e.g., Olver, 1974). In the interval $[0, 2 - n_j]$, the function (A.6) has a minimum at the point

$$z_0 = \tfrac{1}{3}[C(n_j) - 1] \tag{A.7}$$

where $C(n_j)$ is given by (A.1). Hence the expression for the entropy of a nonequilibrium gas (A.3) for the particles obeying 2-statistics, after neglecting the terms which are small compared to G_j, takes the form

$$S = \textstyle\sum_j G_j f(z_0)$$

or

$$S = -\sum_j G_j \left\{ \frac{7 - 3n_j - C(n_j)}{6} \ln\left(\frac{7 - 3n_j - C(n_j)}{6}\right) + \frac{C(n_j) - 1}{3} \ln\left(\frac{C(n_j) - 1}{3}\right) + \frac{1 + 3n_j - C(n_j)}{6} \ln\left(\frac{1 + 3n_j - C(n_j)}{6}\right) \right\} \tag{A.8}$$

It is straightforward to show, by using the identities

$$n_j[7 - 3n_j - C(n_j)] = [C(n_j) - 1][C(n_j) + 1 - n_j]$$

$$[1 + 3n_j - C(n_j)][C(n_j) + 1 - n_j] = 4n_j[C(n_j) - 1]$$

that the expression for the entropy of a nonequilibrium gas obeying 2-statistics resulting from the Boltzmann approach (A.8) agrees with the expression (A.2) which is obtained from the general formula (92). It should be taken into account that the exact coincidence of (A.2) with (A.8) is achieved by means of the correspondence $\sum_i (\cdots) \to \sum_j G_j(\cdots)$, which must be used because the quantities labeled by i are attributed to one quantum state, but the quantities labeled by j relate to G_j states of near energies.

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